Margins and Credit Risk on Vega

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Contents

1 Introduction

The primary financial risk facing a Vega network is credit risk. On a platform where counterparties may be identified by no more than a public key, there is no recourse in the event that a trader owes more in settlement than their posted collateral. It is therefore essential that the protocol be designed to constantly maintain effective collateralisation for all positions.

On a traditional exchange, the biggest risk is that prices move in such a way that a participant is unable to meet her / his obligations (she / he defaults). This exposes the exchange to a loss that has to be covered by the operator or shared amongst the remaining solvent participants. Traditional exchanges keep *margins*: participants are required to post *collateral* in form of cash or securities so that with very high probability any market moves will not expose the exchange to a loss.

On an anonymous distributed exchange the risk is amplified since a participant may choose to leave whenever her / his liabilities exceed the posted margin. Indeed: as she / he is anonymous, leaving the network carries no loss in terms of reputation. Moreover she / he can re-join the exchange under a new identity and carry on, thus maintaining all her / his benefits.

1.1 Financial derivatives

On Vega, a wide range of financial derivates[1] can be described by Vega *smart products*. These are a special type of smart contract designed to allow creation financial derivatives from a toolkit of standard features and economic primitives. See [5]. In particular, given appropriate market inputs (e.g. prices of certain asset at required dates) a smart product will calculate all the cash-flows required for settlement.

Before we can talk about the risk of a financial derivative we need to be able to calculate the probability distribution (within some appropriate stochastic model) of future prices of such a derivative. Any financial derivative can be priced with an appropriate model using arbitrage based arguments. The basic idea is this: say we have an arbitrage-free model[2] of some assets traded assets. Say we have a derivative with a price based on these assets (e.g. our model can contain a risky asset representing some stock and our derivative may be a forward or a call option). The idea is to find a price for that derivative such that if the model allows trading in the derivative asset then it stays free of arbitrage, see e.g. [10].

1.2 Margin versus credit risk

A Vega market participant who has traded a financial derivative is exposing the network to credit risk in case the current price of the derivative is below the price the trade has been made (he / she incurred a loss). We must assume that in this situation, in the absence of other measures, a rational trader will leave the Vega network, assume a new identity and try his / her luck again.

To prevent this, the Vega network will ask traders to post collateral up to a certain margin level. If we were in a situation where the margin can be adjusted in continuous time (i.e. all the time) then we would simply require that the margin amount exceeds the sum of the present value of all the liabilities arising from derivative trades a participant has on the exchange.

However the margin amounts can only be updated at discrete time points (for a traditional exchange this is daily, a blockchain based exchange can re-calculate much more often but finally still only at discrete time-steps). Thus we need to answer the following: what is a reasonable worst case scenario for how much the price of a derivative contract can change in one discrete

[1] The term "derivative" indicates that the cashflows arising in such a contract are determined (derived) from prices / exchange rates / interest rates that can be observed in other markets.

[2] We know that in practice financial markets are not free of arbitrage. The logic of arbitrage-free pricing relies on the idea that any arbitrage that may exist is discovered and exploited by traders so quickly that it doesn't impact the derivative with a much longer time-scale.

fixed time step (call this $\tau > 0$)? We wish our margin amount to exceed the current derivative value plus this worst case scenario to protect the other exchange participants. We will use *coherent risk measures* to account for this worst case scenario.

1.3 Other financial risks

The aim of this document is to describe how Vega will calculate margins required to protect against the credit risk traders create on the exchange. There are other financial risks that arise and are beyond the scope of the document. These are in particular *liquidity risk*, *model risk* and *calibration risk*. We will briefly describe them below.

Consider first the *liquidity risk* which arises in the following situation. Our margin calculation determines that a participant has exceeded their margin (i.e. their posted margin is below minimum margin). The exchange will wish to close out the trade by executing an opposite one (so that the non-defaulting trader is not affected by being made counterparty to this new trade).

If the notional of the trade exceeds anything that's on the order book then there is a problem: we don't have enough liquidity to execute the close-out trade. In practice Vega will mitigate this risk by obliging market makers to provide sufficient liquidity, see [14]. In the rare cases where there is no way to close out the trade, a position-resolution algorithm will be employed, again see [14].

The *model risk* arises as follows. The model is not the world but only an approximation[3]. Model risk attempts to captures the gap between reality and the model. This risk is hard to quantify: to quantify such risk we would ideally like a better model but then why not just price with this better model?

The *calibration risk* arises from whichever calibration procedure is used to obtain appropriate model parameters. Model calibration relies on obtaining market data (historical time series data or current market prices of other derivatives) and using these to choose model parameters so that the model represents current market conditions as well as possible. Calibration relies on appropriate choices: which time-window is to be used for calibration from historical data? Should we give recent data more weight? If we use other market observables to calibrate the model how do we choose which ones to use? Again *calibration risk* is inherently difficult to quantify. Obtaining a consensus on correct calibration in a distributed system is a problem that can only be solved with appropriate economic incentives, see [16].

1.4 Organisation of this document

Section 1 is the introduction, as the reader has seen. In Section 2 we briefly cover Expected Shortfall as an example of "Coherent Risk Measures" and establish some of its basic properties we will need later. In Section 3 we explain how we use expected shortfall to set the margin requirement. Section 4 explains how we will be able to run these type of calculations efficiently with arbitrary smart products defined in terms of a smart product language. Finally, Section 5 covers the two basic risk models currently implemented. Note that implementing others, like stochastic volatility models, is straightforward. In the Appendix A we collect some useful calculations.

2 Expected Shortfall as Coherent Risk Measure

We start by briefly explaining what are the desirable properties of a *risk measure*. In general a *risk measure* is a function, say ρ, which assigns a real-number to a random variable. We think of the random variable X as a payout from a portfolio[4]. A "good" risk measure will have the following

[3] Famously, George Box is quoted as saying: "The most that can be expected from any model is that it can supply a useful approximation to reality: All models are wrong; some models are useful."

[4] Some texts on the subject are written from the point of view of thinking about losses, this means that reading those signs etc. are flipped!

properties:

i) *Monotonicity*: if Y is another portfolio payout s.t. $X \leq Y$ then $\rho(X) \geq \rho(Y)$. So the portfolio that always pays less is deemed more risky.

ii) *Cash invariance:* if $m \in \mathbb{R}$ then $\rho(X+m) = \rho(X) - m$. So adding a non-random cash amount to a portfolio reduces the risk exactly by that amount.

iii) *Positive homogenienty:* if $c > 0$ then $\rho(cX) = c\rho(X)$. If we e.g. double our portfolio then the risk also doubles.

iv) *Subadditivity:* if Y is another portfolio payout then $\rho(X+Y) \leq \rho(X) + \rho(Y)$. This means that a diversified portfolio $X+Y$ will have risk that does not exceed that of X and Y taken separately.

A risk measure satisfying all the above will be called *coherent*, see [7].

We wish to consider *Expected shortfall* as a risk measure. This is sometimes known as *Average value at risk* or *tail value at risk* or *conditional value at risk*. As it is very closely related to the idea of *Value at risk*, we consider that first.

2.1 Value at risk

We fix a level $\alpha > 0$ small and consider X representing a payoff from a portfolio. *Value at risk* is defined as the minimum amount x such that $\mathbb{P}(X + x < 0)$ is smaller than α. This is

$$\mathrm{VaR}_\alpha(X) := \inf\{x \in \mathbb{R} : \mathbb{P}(X + x \leq 0) \leq \alpha\}\,.$$

Note that there are other, effectively equivalent, definitions.[5]

We will write F_X to denote the distribution function[6] of X i.e. $F_X(x) := \mathbb{P}(X \leq x)$. Note that if X is a continuous r.v. then (with F_X^{-1} denoting the inverse of the distribution of X)

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} : -x \leq F_X^{-1}(\alpha)\} = \inf\{x \in \mathbb{R} : x \geq -F_X^{-1}(\alpha)\} = -F_X^{-1}(\alpha)\,,$$

since the inverse of an increasing function is increasing. We have *monotonicity*:

$$\text{If } X \leq Y \text{ then } \mathrm{VaR}_\alpha(X) \geq \mathrm{VaR}_\alpha(Y).$$

We have *cash invarience*:

$$\text{If } m \in \mathbb{R} \text{ then } \mathrm{VaR}_\alpha(X+m) = \mathrm{VaR}_\alpha(X) - m.$$

We have *positive homogeniety*:

$$\text{If } c > 0 \text{ is a constant then } \mathrm{VaR}_\alpha(cX) = \inf\left\{x : \mathbb{P}\left[X + \frac{x}{c} \leq 0\right] \leq \alpha\right\} = c\mathrm{VaR}_\alpha(X).$$

2.2 Problems with VaR

The subadditivity property does not hold for VaR. This means that VaR does not correctly measure the benefits of diversification and one should not ever sum up VaR values for different positions as it has no meaning.

Moreover VaR only tells me how much capital to set aside to have some tolerably small probability of my portfolio being so negative as to the loss exceeding all this capital in a bad situation. What VaR does not capture is the following: given that things are bad and the portfolio plus capital are negative, *how much* do I actually expect to loose?

[5] In particular the definition in [9] which is

$$\overline{\mathrm{VaR}}_p(-X) := \inf\{x \in \mathbb{R} : \mathbb{P}(-X \leq x) \geq p\} = \mathrm{VaR}_\alpha(X)$$

for $\alpha = 1 - p$.

[6] Also known as the cumulative distribution function

2.3 Expected shortfall

The Average Value at Risk / Expected Shortfall for a r.v. X representing payoff of portfolio and $\lambda \in (0, 1)$ is

$$\mathrm{ES}_\lambda(X) := \frac{1}{\lambda}\int_0^\lambda \mathrm{VaR}_\alpha(X)\,d\alpha\,. \tag{1}$$

If X is an absolutely continuous r.v.[7] then

$$\begin{aligned}\mathrm{ES}_\lambda(X) &= -\frac{1}{\lambda}\int_0^\lambda F_X^{-1}(\alpha)\,d\alpha = -\frac{1}{\lambda}\int_{F_X^{-1}(0)}^{F_X^{-1}(\lambda)} z\,dF_X(z)\\ &= \frac{1}{\lambda}\int_{-\infty}^{F_X^{-1}(\lambda)} -z\,dF_X(z) = \frac{1}{F_X(F_X^{-1}(\lambda))}\mathbb{E}\left[-X\mathbb{1}_{X<F_X^{-1}(\lambda)}\right]\\ &= \frac{1}{\mathbb{P}(X<F_X^{-1}(\lambda))}\mathbb{E}\left[-X\mathbb{1}_{X<F_X^{-1}(\lambda)}\right]\end{aligned} \tag{2}$$

and so

$$\mathrm{ES}_\lambda(X) = \mathbb{E}\big[-X\,\big|\,X < -\mathrm{VaR}_\lambda(X)\big]\,.$$

The expected shortfall tells us what is the expected loss given that we have breached the VaR level. If the distribution of X is not continuous then

$$\mathrm{ES}_\lambda(X) = -\frac{1}{\lambda}\Big(\mathbb{E}\left[X\mathbb{1}_{X\le-\mathrm{VaR}_\lambda(X)}\right] + \mathrm{VaR}_\lambda(X)\left(\mathbb{P}\left[X\le-\mathrm{VaR}_\lambda(X)\right]-\lambda\right)\Big) \tag{3}$$

Indeed we note that for X continuous $\mathbb{P}\left[X\le-\mathrm{VaR}_\lambda(X)\right] = \mathbb{P}\left[X\le F_X^{-1}(\lambda)\right] = \lambda$ and so the additional term is simply 0.

We have *monotonicity*:

$$\text{If } X\le Y \text{ then } \mathrm{ES}_\lambda(X)\ge\mathrm{ES}_\lambda(Y).$$

Indeed, due to monotonicity of VaR,

$$\mathrm{ES}_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda \mathrm{VaR}_\alpha(X)\,d\alpha \ge \frac{1}{\lambda}\int_0^\lambda \mathrm{VaR}_\alpha(Y)\,d\alpha = \mathrm{ES}_\lambda(Y)$$

We have *cash invarience*: If $m\in\mathbb{R}$ then

$$\mathrm{ES}_\lambda(X+m) = \frac{1}{\lambda}\int_0^\lambda (\mathrm{VaR}_\alpha(X+m))\,d\alpha = \mathrm{ES}_\lambda(X) - m\,.$$

We have *positive homogeniety*: if $c>0$ is constant then

$$\mathrm{ES}_\lambda(cX) = \frac{1}{\lambda}\int_0^\lambda (\mathrm{VaR}_\alpha(cX))\,d\alpha = c\mathrm{ES}_\lambda(X)\,.$$

Finally, expected shortfall is also *subadditive*: for r.v.s X and Y we have

$$\mathrm{ES}_\lambda(X+Y)\le\mathrm{ES}_\lambda(X)+\mathrm{ES}_\lambda(Y)\,.$$

This is a key property that VaR doesn't posses and this is what makes expected shortfall (together with positive homogeniety) *coherent*. Proof of subadditivity can be found in [9].[8] Expected shortfall can be estimated with Monte-Carlo simulation. Say $(x_i)_{i\in\mathbb{N}}$ are independent samples from

[7] The distribution of an absolutely continuous r.v. has a density with respect to the Lebesgue measure.

[8] Recall that [9] define value-at-risk differently, see footnote 5. They define expected shortfall as

$$\overline{\mathrm{ES}}_p(-X) := \frac{1}{1-p}\int_p^1 \overline{\mathrm{VaR}}_q(-X)\,dq\,.$$

Then, using footnote 5 for first equality and change of variable for the second one and taking $p = 1-\lambda$, we get

$$\overline{\mathrm{ES}}_p(-X) = \frac{1}{1-p}\int_p^1 \overline{\mathrm{VaR}}_{1-q}(X)\,dq = \frac{1}{\lambda}\int_0^\lambda \mathrm{VaR}_\alpha(X)\,d\alpha = \mathrm{ES}_\lambda(X)\,.$$

the distribution of X. We can use expression (3) to devise a Monte–Carlo estimator. Thus we see that we first need an empirical estimate for $\mathrm{VaR}_\lambda(X)$. This is just the $100\cdot\lambda$-percentile of $(x_i)_{i\in\mathbb{N}}$ and we denote this VaR_λ^N. Then

$$\mathrm{ES}_\lambda(X) \approx -\frac{1}{\lambda}\left[\frac{1}{N}\sum_{i=1}^N \left(x_i \mathbb{1}_{\, x_i<-\mathrm{VaR}_\lambda^N}\right) + \mathrm{VaR}_\lambda^N \left(\sum_{i=1}^N \mathbb{1}_{\, x_i<-\mathrm{VaR}_\lambda^N} - \lambda\right)\right]. \tag{4}$$

However this is extremely inefficient because most of the samples are not going to be in the tail of the distribution i.e. most samples will be $x_i \geq -\mathrm{VaR}_\lambda^N$. This means that most samples will be rejected. One way to overcome this is to use a technique called *importance sampling*, see [6]. It may be efficient to base a Monte-Carlo estimate on this equivalent expression for expected shortfall, see [9] for derivation:

$$\mathrm{ES}_\lambda(X) = \inf_{v\in\mathbb{R}}\left\{v + \frac{1}{\lambda}\mathbb{E}[(-X-v)_+]\right\}. \tag{5}$$

The Monte-Carlo estimate can thus be obtained via a one dimensional optimisation problem:

$$\mathrm{ES}_\lambda(X) \approx \inf_{v\in\mathbb{R}}\left\{v + \frac{1}{\lambda}\frac{1}{N}\sum_{i=1}^n[(-x_i-v)_+]\right\}.$$

3 Margins based on Expected shortfall

Our model will be based on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ (that typically supports an n-dimensional Wiener process (Brownian motion) $W=(W_t)_{t\in[0,T]}$). The measure $\mathbb{P}$ is the real-world measure.

3.1 Abstract calculation

Given that we have an arbitrage free model for pricing our derivatives we have at least one risk-neutral measure $\mathbb{Q}$. See e.g. [10]. The current price of a derivative contract which pays X at time T is[9]

$$p_t = B_t\mathbb{E}^{\mathbb{Q}}\left[\frac{X}{B_T}\Big|\mathcal{F}_t\right]. \tag{6}$$

Here B_t is the price of our risk-free asset and $\mathcal{F}_t$ is the σ-algebra representing all the information about assets in the model known at time t.

As we mentioned already if margin amounts were adjusted continuously (and ignoring model risk) we would take the margin at time t, denoted m_t to be simply equal to $-p_t$ (our trader *sold* the derivative, so positive price is a liability for her / him).

In practice we will only be able to re-evaluate the margin requirement at discrete steps given by $\tau > 0$. Thus we wish the margin to cover the expected shortfall at some level $\lambda\in(0,1)$ i.e.

$$m_t = \mathrm{ES}_\lambda(-p_{t+\tau}).$$

3.2 Monte-Carlo based calculation

Let us say that our model contains assets $S^0, S^1,\ldots,S^m$ and that $S_t^0 = B_t$ i.e. S^0 is the risk-free asset which provides discount factors. The payoff of an contingent claim at time T is a measurable function of the paths of the underlying assets i.e. $X = G((S_t^0)_{t\in[0,T]},(S_t^1)_{t\in[0,T]},\ldots,(S_t^m)_{t\in[0,T]})$. Moreover assume that assets prices are given by Markov processes. Then

$$m_t = \mathrm{ES}_\lambda\left(-S_{t+\tau}^0\mathbb{E}^{\mathbb{Q}}\left[\frac{X}{S_T^0}\Big|S_{t+\tau}^i,\, i=0,\ldots,m\right]\right). \tag{7}$$

[9]The price clearly depends on the risk-neutral measure used. In practice one tries to work with "complete markets" in which case the measure is uniquely determined, or one models a liquid market where the measure is parametrised by market price of risk and this is then calibrated to market data along with other model parameters.

This tells us that we have to generate Monte-Carlo scenarios for $S^i_{t+\tau}$ to be able to estimate the expected shortfall.

Thus for each asset i we generate N samples $s^{i,j}$ from the distribution of $S^i_{t+\tau}$. Note that this must be done in the distribution under the real-world measure $\mathbb{P}$. This will be referred to as the *outer Monte-Carlo simulation*. We can now let

$$P^j_{t+\tau}|s^{i,j} := s^{0,j}\mathbb{E}^{\mathbb{Q}}\left[\frac{X}{S^0_T}\Big|S^i_{t+\tau} = s^{i,j},\, i = 0,\ldots,m)\right]. \tag{8}$$

We calculate $\mathrm{VaR}^N_\lambda|s^{i,j}$ as the $100\cdot\lambda$-percentile of $(p^j_{t+\tau})_{j\in\mathbb{N}}|s^{i,j}$. As in (4) we will have

$$m_t \approx \frac{1}{\lambda}\frac{1}{N}\sum_{j=1}^{N}\left(-p^j_{t+\tau}|s^{i,j}\mathbb{1}_{p^j_{t+\tau}|s^{i,j}<-\mathrm{VaR}^N_\lambda|s^{i,j}}\right). \tag{9}$$

Very often in practice we won't be able to evaluate the conditional expectation in (8), i.e. the conditional price of the derivative, exactly. So we will need an *inner Monte-Carlo simulation*: We will need $\tilde{N}$ samples $(x^j_k|s^{i,j})_{k=1}^{\tilde{N}}$ from the distribution, under $\mathbb{Q}$, of $X|s^{i,j}$ as well as $(s^{0,j}_k|s^{i,j})_{k=1}^{\tilde{N}}$ samples from the distribution, under $\mathbb{Q}$, of $S^0_T|s^{i,j}$. Then

$$p^j_{t+\tau}|s^{i,j} \approx s^{0,j}\frac{1}{\tilde{N}}\sum_{k=1}^{\tilde{N}}\frac{x^j_k|s^{i,j}}{s^{0,j}_k|s^{i,j}}. \tag{10}$$

In practice we need to run a *nested simulation*. Say that N and $\tilde{N}$ are say 10^4. In such case we need 10^8 Monte-Carlo samples. This may be prohibitively expensive. There are a number of techniques to overcome this:

i) Some models lead to closed form formulae for the expected shortfall (7). In this case we avoid Monte-Carlo simulation entirely.

ii) Some models allow closed form approximations for the expected shortfall (7). This may be sufficiently accurate in itself. Even if the approximation doesn't provide sufficient accuracy it may be used to construct an efficient *control variate* for more efficient Monte-Carlo simulation. See Glasserman [6] for more details on control variates.

iii) Replace the nested simulation algorithm with e.g. a regression based approach, see Broadie, Du and Moallemi [8] or a multi-level based approach, see Giles and Haji-Ali [13].

iv) Find other efficient *control variates* for more efficient Monte-Carlo estimation of the inner loop. See e.g. [12] or [15].

v) If (4) is used for the expected shortfall calculation, then we use VaR^N_λ (the MC estimate of value at risk) for λ that is typically small and the sample x_i only contributes if $x_i < -\mathrm{VaR}^N_\lambda$. This indicates that *importance sampling* can be used to reduce the variance in the outer loop.

4 Tradable instruments, smart products, risk models

Vega uses *smart product language* to define all the cash-flows and their optionality in a financial derivative. This specifies a *product*. Providing all the product parameters (e.g. strike price, reference oracle source) create an *instrument*. The key feature of an instrument is that it has to be able to provide all settlement cashflows the derivative product provides. In particular, if $\omega \in \Omega$ represents a particular outcome of the world then at this point we can calculate the payoff[10]

$$X(\omega) = G((S^0_t)_{t\in[0,T]}(\omega), (S^1_t)_{t\in[0,T]}(\omega), \ldots, (S^m_t)_{t\in[0,T]}(\omega)),$$

[10] We may want to think about payoffs in different underlying assets. Here we assume that if the payoff is in different assets then we have the exchange rates between all the assets amongst the inputs so that $X(\omega)$ can be calculated as a value in a given single currency.

	Type	Example
Control variate name	`string`	`ATMOption`
Method name for MC valuation	`string`	`BlackScolesCallMC`
Method name for exact formula	`string`	`BlackScolesPrice`
Additional parameters	`key / value pairs`	`Maturity=1, Strike=100`

Table 1: Control variate specification in tradable instrument.

where G is a function of m paths of various underlying assets and in practice it is specified in terms of a *smart product language*.

A *tradable instrument* on Vega has to specify a risk model with model parameters, (so a stochastic model that can simulate future evolution of price trajectories both under an appropriate risk-neutral measure and under a real-world measure). See [14, Section 3] for details.

Such tradable instrument now contains everything that is needed to evaluate (10) and (9) by simulating future scenarios based on the risk model and its parameters and using the instrument logic (provided ultimately by smart language) to provide all the cash-flows arising in such scenario.

4.1 Control variates

Because of the computational expense of performing a naive nested simulation the tradable instrument can, as part of risk parameters, additionally specify which *control variates* should be used in approximating the derivative price.

The plain Monte-Carlo estimator (10) is then replaced by the corresponding control-variate Monte-Carlo estimator, see Glasserman [6, Ch. 4, Sec. 1] or [11]. It is possible to specify more than one control variate.[11]

Simple derivative products in simple models (e.g. option prices in Black–Scholes model) can be used as effective control variates for more complicated products or for efficient pricing under more realistic models. It is therefore desirable that Vega has a large library of risk models providing analytic or semi-analytic formulae for a variety of derivatives as these will can be used to provide control variates. See Section 5 for examples of analytic and semi-analytic formulae that Vega risk models will provide.

4.2 Importance sampling

If the risk model provides the possibility of simulating paths under a different measure (which may, for example, make rare events more likely), then the risk parameters may specify under which measure to perform the simulation and how to perform a change of measure in the final calculation to convert into the appropriate risk-neutral or real-world measures.

See Glasserman [6, Ch. 4, Sec. 6] for more details on how to find appropriate importance sampling change-of-measures.

5 Models

Different asset classes will require use of different stochastic models. In this section we will give details of the margin calculations in several classical models where a stochastic process models the evolution of the asset price directly.[12]

[11] However empirical estimate of the correlation matrix between the controls and the payoff X needs to be obtained and the number of control variates needs to be kept relatively low.

[12] Such models cannot be used in particular for interest rate or credit derivatives.

A model will be useful for margin calculations is it satisfies the following:

i) The model represents key features of the asset being modelled. In particular, the model can provide fat-tailed distributions of the asset returns.

ii) There is an efficient method to calculate the expected shortfall and hence the minimum margin.

iii) There is an efficient method to calibrate the model to market data.

5.1 Black–Scholes model

The Black–Scholes model is a well known and classical model that would have been encountered by any student of finance. In this model we assume that time-value of money is given by a risk-free asset

$$dB_t = rB_t\,dt\,,\ \ B_0 = 1$$

Take a one dimensional Wiener process (Brownian motion) $W = (W_t)_{t\in[0,T]}$. We assume that in the real world measure we have the risky asset price given by

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t\,,\ \ S_0 = S\,.$$

The main advantage of this model is that it provides explicit formulae for expected shortfall of forwards and good analytic approximation for expected shortfall of call and put option prices. It is also easy to calibrate.

The main disadvantage of this model is that asset returns do not exhibit fat tails and moreover the model cannot capture a key feature of asset being modelled: options on the asset exhibit volatility smile.

Forwards in Black–Scholes model

We consider a forward contract which allows the purchase of one unit of a (risky) asset at a future time $T > t$ for K units of currency. The party that agrees to buy the asset, is taking a long position.

The long forward contract payoff (at time T) is $S_T - K$. The fair present value of this is $p_t = S_t - \frac{B_t}{B_T}K$. We consider a trader who sold such contract (short position).

While no model for risky asset is needed to price forward contracts (the price follows form no-arbitrage reasoning in a model-free way), we do need a model to assess the risk. We have at time $t+\tau$ (with $\tau > 0$ small, deterministic)

$$p_{t+\tau} = S_{t+\tau} - \frac{B_{t+\tau}}{B_T}K\,.$$

The expected shortfall is, due to cash invariance (for the short position),

$$\mathrm{ES}_\lambda(-p_{t+\tau}) = \mathrm{ES}_\lambda(-S_{t+\tau}) - \frac{B_{t+\tau}}{B_T}K\,.$$

We can solve the SDE for (S_t) with Itô's formula to get

$$S_{t+\tau} = S_t \exp\left((\mu - \frac{1}{2}\sigma^2)\tau + \sigma(W_{t+\tau} - W_t)\right)\,.$$

Thus, with positive homogenienty of expected shortfall,

$$\mathrm{ES}_\lambda(-p_{t+\tau}) = S_t\,\mathrm{ES}_\lambda(-X) - \frac{B_{t+\tau}}{B_T}K\,,$$

	Minimum margin formula	Risk factor formula
Long forward	$K_0 - K_t + K_t \cdot$ "risk factor"	$\text{ES}_\lambda(X) + 1$
Short forward	$K_t - K_0 + K_t \cdot$ "risk factor"	$\text{ES}_\lambda(-X) - 1$

Table 2: Minimum margin with $r = 0$. Here K_0 is the trade price, K_t the current market price, $X := \exp(\bar{\mu} + \bar{\sigma}Z)$, with $Z \sim N(0,1)$ and with $\bar{\mu} := (\mu - \frac{1}{2}\sigma^2)\tau$, $\bar{\sigma} := \sigma\sqrt{\tau}$.

where $X := \exp(\bar{\mu} + \bar{\sigma}Z)$, with $Z \sim N(0,1)$ and with $\bar{\mu} := (\mu - \frac{1}{2}\sigma^2)\tau$, $\bar{\sigma} := \sigma\sqrt{\tau}$. We have analytic expression for $\text{ES}_\lambda(-X)$ given by (18).

For a long forward position we have (due to cash invariance, positive homogeneity)

$$\text{ES}_\lambda(p_{t+\tau}) = \text{ES}_\lambda(S_{t+\tau}) + \frac{B_{t+\tau}}{B_T}K = S_t\text{ES}_\lambda(X) + \frac{B_{t+\tau}}{B_T}K\,.$$

Note on current underlying price

In practice we may not have real time information on S_t i.e. the current underlying. Nevertheless the exchange will have an order book for the relevant forwards (or even a current traded exercise price K_t and so we can take $S_t = \frac{B_t}{B_T}K_t$.

Explaining the risk measure in terms of quoted price moves

Consider a trader who entered a long forward at K_0. The current (time t) quoted price is K_t. We wish to see the expected shortfall primarily as

$$\text{ES}_\lambda(p_{t+\tau}) = K_0 - K_t + K_t \cdot \text{"risk factor"}.$$

This way we see that if the price on the market is $K_t > K_0$ then the expected shortfall has gone down and vice versa. We can achieve this view with simple manipulation:

$$\begin{aligned}\text{ES}_\lambda(p_{t+\tau}) &= S_t\text{ES}_\lambda(X) + \frac{B_{t+\tau}}{B_T}K_0 \approx \frac{B_t}{B_T}\left(K_t\text{ES}_\lambda(X) + K_0\right) \\ &= \frac{B_t}{B_T}\left(K_0 - K_t + K_t((\text{ES}_\lambda(X) + 1)\right)\,.\end{aligned}$$

This is, up to discounting, what we wanted and in particular if $r = 0$ we have the above with "risk factor" $:= \text{ES}_\lambda(X) + 1$.

For a short position the calculation is

$$\begin{aligned}\text{ES}_\lambda(-p_{t+\tau}) &= S_t\text{ES}_\lambda(-X) - \frac{B_{t+\tau}}{B_T}K_0 \approx \frac{B_t}{B_T}\left(K_t\text{ES}_\lambda(-X) - K_0\right) \\ &= \frac{B_t}{B_T}\left(K_t - K_0 + K_t((\text{ES}_\lambda(-X) - 1)\right)\,.\end{aligned}$$

This is summarised, for $r = 0$, in Table 2.

Net expected shortfall of a portfolio of forwards

We split the portfolio into positions short and long positions. The trader is short with notional / strike / maturity triples $(N_i, K_i, T_i)_{i=1}^n$. The trader is long with notional / strike / maturity triples $(\bar{N}_i, \bar{K}_i, \bar{T}_i)_{i=1}^{\bar{n}}$. The value of the net short position at time t is

$$P_t = \sum_{i=1}^n p_t^i = \sum_{i=1}^n \left(N_i S_t - N_i \frac{B_t}{B_{T_i}}K_i\right)$$

and the value of the net long position is

$$\bar{P}_t = -\sum_{i=1}^{\bar{n}} p_t^i = -\sum_{i=1}^{\bar{n}} \left(\bar{N}_i S_t - \bar{N}_i \frac{B_t}{B_{\bar{T}_i}} \bar{K}_i \right) .$$

Thus

$$\begin{aligned} \mathrm{ES}_\lambda(-P_{t+\tau} - \bar{P}_{t+\tau}) &= \mathrm{ES}_\lambda \left(-\sum_{i=1}^{n} N_i S_{t+\tau} + \sum_{i=1}^{n} N_i \frac{B_{t+\tau}}{B_{T_i}} K_i + \sum_{j=1}^{\bar{n}} \bar{N}_j S_{t+\tau} - \sum_{j=1}^{\bar{n}} \bar{N}_j \frac{B_{t+\tau}}{B_{\bar{T}_i}} \bar{K}_i \right) \\ &= \mathrm{ES}_\lambda(\alpha S_{t+\tau}) + \beta, \end{aligned}$$

where

$$\alpha := -\sum_{i=1}^{n} N_i + \sum_{j=1}^{\bar{n}} \bar{N}_j \quad \text{and} \quad \beta := -\sum_{i=1}^{n} N_i \frac{B_{t+\tau}}{B_{T_i}} K_i + \sum_{j=1}^{\bar{n}} \bar{N}_j \frac{B_{t+\tau}}{B_{\bar{T}_i}} \bar{K}_i .$$

Hence

$$\mathrm{ES}_\lambda(-P_{t+\tau} - \bar{P}_{t+\tau}) = \mathrm{sgn}(\alpha)\alpha S\, \mathrm{ES}_\lambda(\mathrm{sgn}(\alpha) X) + \beta .$$

Here $X := \exp(\bar{\mu} + \bar{\sigma} Z)$, with $Z \sim N(0,1)$ and with $\bar{\mu} := (\mu - \frac{1}{2}\sigma^2)\tau$, $\bar{\sigma} := \sigma\sqrt{\tau}$ as above. Moreover the expected shortfall is given by (17) or (18) depending on $\mathrm{sgn}(\alpha)$.

European style options in Black–Scholes model

We assume that in the real world measure we have a risk-free asset (bank account)

$$dB_t = rB_t\, dt\,, \quad B_0 = 1$$

and a risky asset

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t\,, \quad S_0 = S\,.$$

There is a unique risk neutral measure $\mathbb{Q}$. If $g : [0,\infty) \to \mathbb{R}$ is a function describing the option payoff (so that e.g. a call is $g(S) = [S - K]_+$) then the option price at time $t + \tau$ is given by

$$v_{t+\tau} = B_{t+\tau} \mathbb{E}^{\mathbb{Q}} \left[\frac{g(S_T)}{B_T} \Big| S_{t+\tau} \right] .$$

In the particular case of call option we then have the price from Black–Scholes formula, see (20), i.e.

$$v_{t+\tau} = \mathrm{BSformulaCall}(S_{t+\tau}, K, T - (t+\tau), r, \sigma)\,.$$

There seems to be no closed-form expression for expected shortfall in the Black–Scholes model.

Analytic approximation based on hedging portfolio

In case the risk horizon $\tau > 0$ is short we may derive a useful approximation from the replicating portfolio representation of the option price. Indeed let v be given by

$$v(t,S) := B_t \mathbb{E}^{\mathbb{Q}} \left[\frac{g(S_T)}{B_T} \Big| S_t = S \right] .$$

Then using the "delta-hedging" strategy

$$\begin{aligned} v_{t+\tau} &= v_t + \int_t^{t+\tau} \frac{\partial v}{\partial S}(u, S_u)\, dS_u + \int_t^{t+\tau} \frac{v_u - S_u \frac{\partial v}{\partial S}(u,S_u)}{B_u}\, dB_u \\ &= v_t + \int_t^{t+\tau} S_u \frac{\partial v}{\partial S}(u,S_u)\mu\, du + \int_t^{t+\tau} S_u \frac{\partial v}{\partial S}(u,S_u)\sigma\, dW_u + \int_t^{t+\tau} r[v_u - S_u \frac{\partial v}{\partial S}(u,S_u)]\, du \\ &= v_t + \int_t^{t+\tau} S_u \frac{\partial v}{\partial S}(u,S_u)(\mu - r)\, du + \int_t^{t+\tau} S_u \frac{\partial v}{\partial S}(u,S_u)\sigma\, dW_u + \int_t^{t+\tau} r v_u\, du \end{aligned}$$

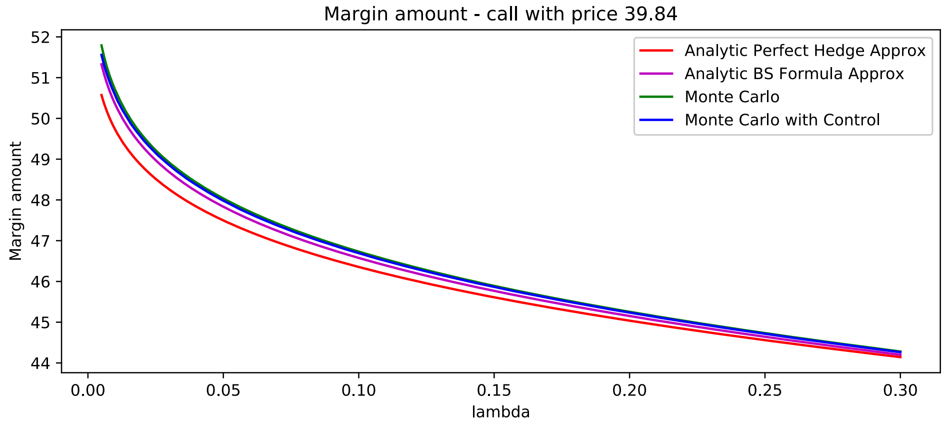

Figure 1: The analytic approximation for expected shortfall is not precise enough but it is very good basis for control variate. The control variate estimator used $N = 5000$ samples. The plain Monte Carlo used $N = 50\,000$.

If we assume that $\tau > 0$ is small then we can approximate this as

$$v_{t+\tau} \approx (1+\tau r)v_t + S_t \frac{\partial v}{\partial S}(t, S_t)(\mu - r)\,\tau + S_t \frac{\partial v}{\partial S}(t, S_t)\sigma\sqrt{\tau} Z\,, \tag{11}$$

where $Z \sim N(0, 1)$.

We now consider the expected shortfall for a trader who sold such option. From (11) and with cash invariance

$$\begin{aligned} \mathrm{ES}_\lambda(-v_{t+\tau}) &\approx \mathrm{ES}_\lambda\left(-(1+\tau r)v_t - S_t \frac{\partial v}{\partial S}(t, S_t)(\mu - r)\,\tau - S_t \frac{\partial v}{\partial S}(t, S_t)\sigma\sqrt{\tau} Z\right) \\ &= (1+\tau r)v_t + S_t \frac{\partial v}{\partial S}(t, S_t)(\mu - r)\,\tau + \mathrm{ES}_\lambda\left(-S_t \frac{\partial v}{\partial S}(t, S_t)\sigma\sqrt{\tau} Z\right) \end{aligned}$$

If $\frac{\partial v}{\partial S}(t, S_t) \geq 0$ then positive homogeniety leads to ($S_t \sigma \sqrt{\tau} > 0$ always):

$$\mathrm{ES}_\lambda\left(S_t \frac{\partial v}{\partial S}(t, S_t)\sigma\sqrt{\tau}(-Z)\right) = S_t \frac{\partial v}{\partial S}(t, S_t)\sigma\sqrt{\tau}\mathrm{ES}_\lambda\left(-Z\right)\,.$$

If $\frac{\partial v}{\partial S}(t, S_t) < 0$ then

$$\mathrm{ES}_\lambda\left(S_t \frac{\partial v}{\partial S}(t, S_t)\sigma\sqrt{\tau}(-Z)\right) = -S_t \frac{\partial v}{\partial S}(t, S_t)\sigma\sqrt{\tau}\mathrm{ES}_\lambda(Z)\,.$$

Since for $Z \sim N(0, 1) \sim -Z$ we have $\mathrm{ES}_\lambda(Z) = \mathrm{ES}_\lambda(-Z)$, we finally obtain

$$\begin{aligned} \mathrm{ES}_\lambda(-v_{t+\tau}) \approx& (1+\tau r)v_t + S_t \frac{\partial v}{\partial S}(t, S_t)(\mu - r)\,\tau \\ &+ \mathrm{sgn}\left(\frac{\partial v}{\partial S}(t, S_t)\right) S_t \frac{\partial v}{\partial S}(t, S_t)\sigma\sqrt{\tau}\mathrm{ES}_\lambda\left(Z\right)\,. \end{aligned}$$

Note that this approximation is has a tendency to notably underestimate the expected shortfall, see Figure 1. The approximation is useful as control variate in Monte-Carlo simulation.

Analytic approximation based on the Black–Scholes formula

We know that

$$\begin{aligned} v_{t+\tau} &= S_{t+\tau} P_1(S_{t+\tau}, K, T-t, r, \sigma) - e^{-r(T-(t+\tau))} K P_2(S_{t+\tau}, K, T-t, r, \sigma) \\ &\approx S_{t+\tau} P_1(S_t, K, T-t, r, \sigma) - e^{-r(T-t)} K P_2(S_t, K, T-t, r, \sigma)\,. \end{aligned}$$

Keeping K, $T-t$, r, σ fixed in what follows, writing $P_i(S_t) := P_i(S_t, K, T-t, r, \sigma)$, and taking $X \sim e^{(\mu - \frac{1}{2}\sigma^2)\tau + \tau\sigma Z}$ with $Z \sim N(0,1)$, we get

$$\mathrm{ES}_\lambda(-v_{t+\tau}) \approx S_t P_1(S_t)\mathrm{ES}_\lambda(-X) - e^{-r(T-t)} K P_2(S_t)\,,$$

i.e. we just need the expected shortfall of negative lognormal r.v.

Explaining the risk measure in terms of quoted price moves

In fact on the exchange when the trade is entered, no money changes hands and all is handled at settlement. So the minimum margin for *short call* should in fact be

$$m_t = \mathrm{ES}_\lambda(-v_{t+\tau} + e^{rt}v_0) = \mathrm{ES}_\lambda(-v_{t+\tau}) - e^{rt}v_0\,.$$

And so

$$\begin{aligned} m_t &\approx S_t P_1(S_t)\mathrm{ES}_\lambda(-X) - e^{-r(T-t)} K P_2(S_t) - e^{rt}v_0 \\ &= v_t - e^{rt}v_0 + S_t P_1(S_t)\left(\mathrm{ES}_\lambda(-X) - 1\right)\,. \end{aligned}$$

For someone who bought a call i.e. for a *long call* this is

$$\begin{aligned} m_t &= \mathrm{ES}_\lambda(v_{t+\tau} - e^{rt}v_0) = \mathrm{ES}_\lambda(v_{t+\tau}) + e^{rt}v_0 \\ &\approx S_t P_1(S_t)\mathrm{ES}_\lambda(X) + e^{-r(T-t)} K P_2(S_t) + e^{rt}v_0 \\ &= e^{rt}v_0 - v_t + S_t P_1(S_t)\left(\mathrm{ES}_\lambda(X) + 1\right)\,. \end{aligned}$$

If we write p_t for the time t price of a call option then for a *short put* we have

$$m_t = \mathrm{ES}_\lambda\left(-p_{t+\tau} + e^{rt}p_0\right) = \mathrm{ES}_\lambda\left(-p_{t+\tau}\right) - e^{rt}p_0\,.$$

From (22) we get

$$\begin{aligned} m_t &= \mathrm{ES}_\lambda\left(S_{t+\tau}(1 - P_1(S_{t+\tau})) - Ke^{-r(T-(t+\tau))}(1 - P_2(S_{t+\tau}))\right) - e^{rt}p_0 \\ &\approx \mathrm{ES}_\lambda\left(S_{t+\tau}(1 - P_1(S_t)) - Ke^{-r(T-t)}(1 - P_2(S_t)\right) - e^{rt}p_0 \\ &= S_t(1 - P_1(S_t))\mathrm{ES}_\lambda(X) + Ke^{-r(T-t)}(1 - P_2(S_t)) - e^{rt}p_0 \\ &= p_t - e^{rt}p_0 + S_t(1 - P_1(S_t))\left(\mathrm{ES}_\lambda(X) + 1\right)\,. \end{aligned}$$

Finally for a *long put* we have

$$m_t = \mathrm{ES}_\lambda\left(p_{t+\tau} - e^{rt}p_0\right) = \mathrm{ES}_\lambda\left(p_{t+\tau}\right) + e^{rt}p_0\,.$$

Then

$$\begin{aligned} m_t &\approx \mathrm{ES}_\lambda\left(Ke^{-r(T-t)}(1 - P_2(S_t)) - S_{t+\tau}(1 - P_1(S_t))\right) + e^{rt}p_0 \\ &= S_t(1 - P_1(S_t))\mathrm{ES}_\lambda(-X) - Ke^{-r(T-t)}(1 - P_2(S_t) + e^{rt}p_0 \\ &= e^{rt}p_0 - p_t + S_t(1 - P_1(S_t))\left(\mathrm{ES}_\lambda(-X) - 1\right)\,. \end{aligned}$$

This is summarised, for $r = 0$, in Table 3.

5.2 Black–Scholes Model with Jumps

In the Black–Scholes model with jumps we again assume that time-value of money is given by a risk-free asset

$$dB_t = rB_t\,dt\,,\quad B_0 = 1$$

	Minimum margin formula	Risk factor formula
Short call	$v_t - v_0 + S_t \cdot$ "risk factor"	$P_1(S_t)\,(\mathrm{ES}_\lambda(-X) - 1)$
Long call	$v_0 - v_t + S_t \cdot$ "risk factor"	$P_1(S_t)\,(\mathrm{ES}_\lambda(X) + 1)$
Short put	$p_t - p_0 + S_t \cdot$ "risk factor"	$(1 - P_1(S_t))\,(\mathrm{ES}_\lambda(X) + 1)$
Long put	$p_0 - p_t + S_t \cdot$ "risk factor"	$(1 - P_1(S_t))\,(\mathrm{ES}_\lambda(-X) - 1)$

Table 3: Minimum margin with $r = 0$ in Black–Scholes model. Here v_0, p_0 is the trade price for call and put respectively, v_t, p_t the current market prices for call and put respectively, S_t is the underlying price, $X := \exp(\bar{\mu} + \bar{\sigma} Z)$, with $Z \sim N(0,1)$ and with $\bar{\mu} := (\mu - \frac{1}{2}\sigma^2)\tau$, $\bar{\sigma} := \sigma\sqrt{\tau}$. Moreover $P_1(S_t) = P_1(S_t, K, T-t, r, \sigma)$ is given in (21).

The risky asset price is given by[13]

$$dS_t = \mu S_{t-}\,dt + \sigma S_{t-}\,dW_t + S_{t-}(dJ_t - \alpha\gamma\,dt)\,,\ \ S_0 = S\,.$$

Here $W = (W_t)_{t\in[0,T]}$ is a one dimensional Wiener process (Brownian motion) under the real-world measure $\mathbb{P}$ and the pure jump process J is given by *compound Poisson process*

$$J_t = \sum_{j=1}^{N_t} (e^{Z_j} - 1)\,,$$

with some *jump times* $0 < \tau_1 < \tau_2 < \cdots$ and $N_t := \sup\{n : \tau_n \leq t\}$. We will assume that N is a Poisson process independent of W with rate γ i.e. that the *inter arrival times* $\tau_{j+1} - \tau_j$ follow exponential distribution with this γ so that $\mathbb{P}(\tau_{j+1} - \tau_j \leq u) = 1 - e^{-\gamma u}$. It is possible to assume various distributions for Z_j but the simplest reasonable assumption is that they are mutually independent, independent of the inter arrival times and of W and and identically distributed with $\alpha := \mathbb{E}\left[e^{Z_j} - 1\right]$ the average jump size. Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by W and J. Due to this and the independence of increments of J_t we have, for $t > s$,

$$\mathbb{E}\left[J_t - t\gamma\alpha|\mathcal{F}_s\right] = \mathbb{E}\left[J_t - J_s|\mathcal{F}_s\right] + J_s - t\gamma\alpha = \mathbb{E}\left[J_t - J_s\right] + J_s - t\gamma\alpha = J_s - s\gamma\alpha$$

and so the process given by $J_t - t\gamma\alpha$ is a martingale.

For $t \in [\tau_{j-1}, \tau_j)$ we have

$$S_t = S_{\tau_{j-1}} \exp\left[\left(\mu - \frac{1}{2}\sigma^2 - \alpha\gamma\right)(t - \tau_{j-1}) + \sigma(W_t - W_{\tau_{j-1}})\right]$$

as for geometric brownian motion without jumps. Then at $t = \tau_j$

$$\Delta S_{\tau_j} = S_{\tau_j} - S_{\tau_j-} = S_{\tau_j-}(J_{\tau_j} - J_{\tau_j-}) = S_{\tau_j-}(e^{Z_j} - 1)$$

and so $S_{\tau_j} = S_{\tau_j-}e^{Z_j}$ which means that

$$S_{\tau_j} = S_{\tau_{j-1}} \exp\left[\left(\mu - \frac{1}{2}\sigma^2 - \alpha\gamma\right)(\tau_j - \tau_{j-1}) + \sigma(W_t - W_{\tau_{j-1}})\right]\exp(Z_j)\,.$$

Recursively, we then get that for any $t \geq 0$

$$S_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2 - \alpha\gamma\right)t + \sigma W_t\right]\prod_{j=1}^{N_t}\exp(Z_j)\,.$$

We can write $S_t = S_0 e^{X_t}$ where

$$X_t = \left(\mu - \frac{1}{2}\sigma^2 - \alpha\gamma\right)t + \sigma W_t + \sum_{j=1}^{N_t} Z_j\,.$$

[13] As usual $S_{t-} := \lim_{u\nearrow t} S_u$.

Characteristic Function

There are no explicit formulae for e.g. options prices or expected shortfall in the jump-diffusion model. However, efficient calculations are possible since we can calculate the characteristic function[14] of X_t exactly.

Due to the independence assumption

$$\phi_{X_t}(u) = \mathbb{E}\left[e^{iu\left(\left(\mu-\frac{1}{2}\sigma^2-\alpha\gamma\right)t+\sigma W_t\right)}\right]\mathbb{E}\left[e^{iu\sum_{j=1}^{N_t}Z_j}\right].$$

Now with the tower property and conditional independence of Z_j's we have

$$E\left[e^{iu\sum_{j=1}^{N_t}Z_j}\right] = \mathbb{E}\left[\mathbb{E}\left[\prod_{j=1}^{N_t}e^{iuZ_j}\right]\Big|N_t\right] = \mathbb{E}\left[\phi_Z(u)^{N_t}\right] = \sum_{n=0}^{\infty}\phi_Z^n(u)\frac{\gamma^n t^n}{n!}e^{-\gamma t} = e^{\gamma t(\phi_Z(u)-1)},$$

where ϕ_Z denotes the characteristic function of Z_j's. Hence[15]

$$\phi_{X_t}(u) = \exp\left[\left(\mu-\frac{1}{2}\sigma^2-\alpha\gamma\right)itu-\frac{1}{2}\sigma^2tu^2\right]\exp\left[\gamma t(\phi_Z(u)-1)\right]. \tag{12}$$

Risk-neutral Measures and Incompleteness

If we assume that B and S are the only two assets available in this model then it is incomplete. Indeed, define

$$Z_t = e^{(\gamma-\tilde{\gamma})t}\prod_{j=1}^{N_t}\frac{\tilde{\gamma}}{\gamma}e^{-\varphi W_t-\frac{1}{2}\varphi^2 t}.$$

It can be shown that Z_t is a $\mathbb{P}$-martingale and as a consequence of Girsanov's theorem, under a new measure $d\mathbb{Q} = Z_T\,d\mathbb{P}$ (for some fixed time horizon $T>0$), $J=(J_t)_{t\geq 0}$ is a compound Poisson process with intensity $\tilde{\gamma}$, $W^{\mathbb{Q}} = (W_t^{\mathbb{Q}})_{t\geq 0}$ given by

$$W^{\mathbb{Q}} = W_t + \varphi t$$

is a Wiener process independent of J. Note that the new measure $\mathbb{Q}$ depends on choice of $\tilde{\gamma}>0$ and φ. Now

$$d(e^{-rt}S_{t-}) = S_{t-}e^{-rt}\left[\sigma\,dW_t^{\mathbb{Q}} + dJ_t - \tilde{\gamma}\alpha\,dt\right]$$

as long as we choose $\tilde{\gamma}>0$ and φ such that

$$(\mu-r)+(\tilde{\gamma}-\gamma)\alpha-\sigma\varphi = 0. \tag{13}$$

Since $W^{\mathbb{Q}}$ and $J_t-\tilde{\gamma}\alpha t$ are $\mathbb{Q}$-martingales we know that $e^{-rt}S_t$ is also a $\mathbb{Q}$-martingale but the risk-neutral measure is not unique.[16] This means that contingent claims cannot be replicated. In particular the usual delta-hedging strategy will not protect against jumps.

For simplicity we will assume that $\gamma=\tilde{\gamma}$ so that $\varphi=\frac{\mu-r}{\sigma}$ and we have

$$X_t = \left(r-\frac{1}{2}\sigma^2-\alpha\gamma\right)t+\sigma W_t^{\mathbb{Q}}+\sum_{j=1}^{N_t}Z_j$$

and from (12) we see its characteristic function under $\mathbb{Q}$ is

$$\phi_{X_t}^{\mathbb{Q}}(u) = \exp\left[\left(r-\frac{1}{2}\sigma^2-\alpha\gamma\right)itu-\frac{1}{2}\sigma^2tu^2\right]\exp\left[\gamma t(\phi_Z(u)-1)\right]. \tag{14}$$

[14] For an $\mathbb{R}^d$-valued r.v. X its characteristic function defined as $\phi(u) := \mathbb{E}[e^{iuX}]$. If X has the law μ then we see that $\phi(u) = \int_{\mathbb{R}^d} e^{iux}\,\mu(dx)$ which is exactly the Fourier transform of the law μ. If we wish to emphasise the dependence on X then we will write ϕ_X in place of ϕ.

[15] For $Z\sim N(a,b^2)$ we have $\phi_Z(u) = e^{iau-\frac{1}{2}b^2u^2}$.

[16] Since we are solving one equation, namely (13), with two unknowns.

We now fix the density f of Y_j so that $(1 + Y_j) \sim e^{a+bZ_j}$ for $Z_j \sim N(0,1)$. This means that $1 + Y_j$ are log-normally distributed with mean a and variance b^2 and $\alpha = \mathbb{E}Y_j = e^{a+\frac{1}{2}b^2} - 1$.

Pricing European options with Fourier transform

Once we have fixed the market price of risk and the associated risk neutral measure $\mathbb{Q}$ we have that any contingent claim X has price at time t given by (6). For a call option with strike $K > 0$ this is

$$c_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)_+|\mathcal{F}_t\right] = S_t e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(e^{X_T} - e^k)_+|\mathcal{F}_t\right]$$

where k is such that $\frac{K}{S_t} = e^k$.

Without loss of generality we take $t = 0$ and define

$$C(k) := e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(e^{X_T} - e^k)_+\right].$$

Even though there is no closed-form formula, this quantity can be evaluated efficiently using the discrete Fourier transform, see Carr and Madan [3]. Here we give the main ingredients of the algorithm. Since C_k is not an integrable function of k its Fourier transform is not defined. Instead, we consider

$$z(k) := C(k) - C^{\mathrm{BS}}(k)$$

where C^{BS} is the price of a call option with strike e^k, see Appendix A.5. It can be shown that $k \mapsto z(k)$ is integrable and moreover its Fourier transform is

$$\zeta_T(v) := (Fz)(v) = \int_{\mathbb{R}} e^{ivk} z(k)\, dk = e^{ivrT}\frac{\Phi_T^{(1)}(v-i) - \Phi_T^{(2)}(v-i)}{iv(1+iv)}$$

where $\Phi_T^{(1)}$ is the characteristic function of X_t under $\mathbb{Q}$ so, due to (14), this is

$$\Phi_T^{(1)}(u) = \exp\left\{\left(r - \frac{1}{2}\sigma^2 - \alpha\gamma\right)T\,iu - \frac{1}{2}\sigma^2 T u^2\right\}\exp\left\{\gamma T(\phi_Z(u) - 1)\right\}$$

and where $\Phi_T^{(2)}$ is the characteristic function of the risky asset price process in the Black–Scholes model, so due to (14) with $\gamma = 0$, this is

$$\Phi_T^{(2)}(u) = \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)T\,iu - \frac{1}{2}\sigma^2 T u^2\right\}.$$

Finally, to obtain $C(k)$ we invert the Fourier transform:

$$C(k) = z(k) + C^{\mathrm{BS}}(k) = \frac{1}{2\pi}\int_{\mathbb{R}} \zeta_T(v)e^{-ivk}\, dv + C^{\mathrm{BS}}(k). \tag{15}$$

We note that since $z(k)$ is real we can conclude that $v \mapsto \zeta_T(v)$ is odd in its imaginary part and even in its real part and so $\zeta_T(v)e^{-ivk} = \zeta_T(v)e^{ivk}$ which means it suffices to integrate over the positive real axis:

$$C(k) = \frac{1}{\pi}\int_0^\infty \zeta_T(v)e^{-ivk}\, dv + C^{\mathrm{BS}}(k).$$

If we only want to have the option value for one strike then the inverse of the Fourier transform is best obtained by numerical integration while if we want to know the option price for a range of strikes one should use the Discrete Fourier transform.[17] See Appendix A.7 for more detail.

[17] If, for purposes of numerical integration, we approximate the real numbers by a finite partition of size N then simple numerical integration will need to evaluate ζ_T order of N times. The Discrete Fourier transform needs order of $N \log N$ evaluations.

Forwards in Black–Scholes model with jumps

From Section 5.1 we know that the key quantity that needs to be calculated is

$$\mathrm{ES}_\lambda(-S_{t+\tau}) = S_t \mathrm{ES}_\lambda(-e^{X_{t+\tau}}) \text{ and } \mathrm{ES}_\lambda(S_{t+\tau}) = S_t \mathrm{ES}_\lambda(e^{X_{t+\tau}})\,.$$

Take $T = t + \tau$. We also recall (5) which tells us that we can view the expected shortfall calculation as a 1D minimisation problem. This can be done efficiently as long as we can calculate $\mathbb{E}[(-e^{X_T} - v)_+]$ and $\mathbb{E}[(e^{X_T} - v)_+]$ efficiently.

Let us start with the former. We note that for all $v \geq 0$ we have $\mathbb{E}[(-e^{X_T} - v)_+] = 0$ and so we only need to consider $v = -e^k$ for $k \in \mathbb{R}$ so that $\mathbb{E}[(-e^{X_T} - v)_+] = \mathbb{E}[(e^k - e^{X_T})_+]$.

$$(e^k - e^{X_T})_+ - (e^{X_T} - e^k)_+ = e^k - e^{X_T}$$

Moreover the process $e^{-\mu t} e^{X_t}$ is a $\mathbb{P}$-martingale which means that $\mathbb{E}[e^{X_T}] = e^{\mu T}$ and so

$$\mathbb{E}[(e^k - e^{X_T})_+] = e^k - e^{\mu T} + \mathbb{E}[(e^{X_T} - e^k)_+] = e^k - e^{\mu T} + e^{\mu T} C(k)\,,$$

where $C(k) := e^{-\mu T}\mathbb{E}[(e^{X_T} - e^k)_+]$. This $C(k)$ is identical to European option price (with interest rate parameter μ) in jump-diffusion models and is obtained by (15). So, since $\mathbb{E}[(-e^{X_T} - v)_+] = 0$ for $v \geq 0$ we have

$$\begin{aligned}\mathrm{ES}_\lambda(e^{X_T}) &= \inf_{v\in\mathbb{R}} \left\{ v + \frac{1}{\lambda}\mathbb{E}[(-e^{X_T} - v)_+] \right\} = \min\left[0, \inf_{v\in(-\infty,0)} \left\{ v + \frac{1}{\lambda}\mathbb{E}[(-e^{X_T} - v)_+] \right\}\right] \\ &= \min\left[0, \inf_{k\in\mathbb{R}} \left\{ -e^k + \frac{1}{\lambda}\mathbb{E}[(e^k - e^{X_T})_+] \right\}\right]\,.\end{aligned}$$

Hence

$$\mathrm{ES}_\lambda(e^{X_T}) = \min\left[0, \inf_{k\in\mathbb{R}} \left\{ -e^k + \frac{1}{\lambda}\left(e^k - e^{\mu T} + e^{\mu T} C(k) \right) \right\}\right]\,.$$

Let us look at $\mathbb{E}[(e^{X_T} - v)_+]$. Clearly if $v \leq 0$ then $\mathbb{E}[(e^{X_T} - v)_+] = \mathbb{E}[e^{X_T}] - v = e^{\mu T} - v$. If $v > 0$ then we take $e^k = v$ and get that

$$\mathbb{E}[(e^{X_T} - v)_+] = \mathbb{E}[(e^{X_T} - e^k)_+] = e^{\mu T} C(k)\,.$$

As before $C(k)$ is identical to European option price (with interest rate parameter μ) in jump-diffusion models and is obtained by (15). So

$$\begin{aligned}\mathrm{ES}_\lambda(-e^{X_T}) &= \inf_{v\in\mathbb{R}} \left\{ v + \frac{1}{\lambda}\mathbb{E}[(e^{X_T} - v)_+] \right\} \\ &= \min\left[\inf_{v\in(-\infty,0]} \left\{ v + \frac{1}{\lambda}\left(e^{\mu T} - v\right) \right\}, \inf_{v\in(0,\infty)} \left\{ v + \frac{1}{\lambda}\mathbb{E}[(e^{X_T} - v)_+] \right\} \right]\,.\end{aligned}$$

Hence

$$\mathrm{ES}_\lambda(-e^{X_T}) = \min\left[\frac{1}{\lambda} e^{\mu T}, \inf_{k\in\mathbb{R}} \left\{ e^k + \frac{1}{\lambda} e^{\mu T} C(k) \right\} \right]\,.$$

A Appendix

A.1 Expected shortfall of $Z \sim N(0,1)$

Consider $Z \sim N(0,1)$. We immediately note that since $-Z \sim N(0,1)$ due to symmetry, we have $\mathrm{ES}_\lambda(-Z) = \mathrm{ES}_\lambda(Z)$.

Moreover $\mathrm{VaR}_\alpha(Z) = -F_Z^{-1}(\alpha)$ and, using (2),

$$\begin{aligned}\mathrm{ES}_\lambda(Z) &= \frac{1}{\mathbb{P}(Z < F_Z^{-1}(\lambda))}\mathbb{E}\left[-Z\mathbb{1}_{Z<F_Z^{-1}(\lambda)}\right] = -\frac{1}{\lambda}\mathbb{E}\left[Z\mathbb{1}_{Z<F_Z^{-1}(\lambda)}\right] \\ &= -\frac{1}{\lambda}\int_{-\infty}^{F_Z^{-1}(\lambda)} z\, f_Z(z)\, dz\,,\end{aligned} \tag{16}$$

where f_Z denotes the density of $N(0,1)$.

A.2 Expected shortfall of lognormal r.v.s

Consider a lognormal r.v.

$$X := \exp\left(\bar{\mu} + \bar{\sigma} Z\right)\,,$$

where $Z \sim N(0,1)$. Then

$$\mathrm{ES}_\lambda(X) = -\frac{e^{\bar{\mu}+\frac{\bar{\sigma}^2}{2}}}{\lambda} F_Z\left(F_Z^{-1}(\lambda) - \bar{\sigma}\right)\,. \tag{17}$$

We will now show why this is the case: first we note that

$$F_X^{-1}(\alpha) = \exp\left(\bar{\mu} + \bar{\sigma} F_Z^{-1}(\alpha)\right)\,.$$

By definition of expected shortfall

$$\mathrm{ES}_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda -F_X^{-1}(\alpha)\, d\alpha = -\frac{1}{\lambda}\int_0^\lambda \exp\left(\bar{\mu} + \bar{\sigma} F_Z^{-1}(\alpha)\right)\, d\alpha\,.$$

With a change of variable and writing f_Z for the density of Z we get

$$\mathrm{ES}_\lambda(X) = -\frac{e^{\bar{\mu}}}{\lambda}\int_{-\infty}^{F_Z^{-1}(\lambda)} e^{\bar{\sigma} z} f_Z(z)\, dz\,.$$

Completing the squares and another change of variable leads to

$$\mathrm{ES}_\lambda(X) = -\frac{e^{\bar{\mu}+\frac{\bar{\sigma}^2}{2}}}{\lambda}\int_{-\infty}^{F_Z^{-1}(\lambda)-\bar{\sigma}} e^{-\frac{1}{2}y^2}\, dy = -\frac{e^{\bar{\mu}+\frac{\bar{\sigma}^2}{2}}}{\lambda} F_Z\left(F_Z^{-1}(\lambda) - \bar{\sigma}\right)\,.$$

A.3 Expected shortfall of negative lognormal r.v.s

Consider a negative lognormal r.v.

$$Y := -\exp\left(\bar{\mu} + \bar{\sigma} Z\right)\,,$$

where $Z \sim N(0,1)$. Then

$$\mathrm{ES}_\lambda(Y) = \frac{e^{\bar{\mu}+\frac{\bar{\sigma}^2}{2}}}{\lambda}\left[1 - F_Z\left(F_Z^{-1}(1-\lambda) - \bar{\sigma}\right)\right]\,. \tag{18}$$

We will now show why this is the case: first we note that

$$F_Y^{-1}(\alpha) = -\exp\left(\bar{\mu} + \bar{\sigma} F_Z^{-1}(1-\alpha)\right)\,.$$

By definition of expected shortfall

$$\mathrm{ES}_\lambda(Y) = \frac{1}{\lambda}\int_0^\lambda -F_Y^{-1}(\alpha)\, d\alpha = \frac{1}{\lambda}\int_0^\lambda \exp\left(\bar{\mu} + \bar{\sigma} F_Z^{-1}(1-\alpha)\right)\, d\alpha\,.$$

With a change of variable and writing f_Z for the density of Z we get

$$\mathrm{ES}_\lambda(X) = \frac{e^{\bar{\mu}}}{\lambda} \int_{F_Z^{-1}(1-\lambda)}^{\infty} e^{\bar{\sigma} z} f_Z(z)\, dz\,.$$

Completing the squares and another change of variable leads to

$$\mathrm{ES}_\lambda(X) = \frac{e^{\bar{\mu}+\frac{\bar{\sigma}^2}{2}}}{\lambda} \int_{F_Z^{-1}(1-\lambda)-\bar{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy = \frac{e^{\bar{\mu}+\frac{\bar{\sigma}^2}{2}}}{\lambda} \left[1 - F_Z\left(F_Z^{-1}(1-\lambda) - \bar{\sigma}\right)\right].$$

A.4 Put-Call parity

Say c_t, p_t are the time t call and put price respectively. Then the following identity holds

$$p_t = c_t - S_t + KD(t,T)\,, \tag{19}$$

where $D(t,T)$ is the discount factor between t and T e.g. $D(t,T) = e^{-r(T-t)}$ if we assume constant continuously compounded interest rate.

A.5 Black–Scholes formula

For current risky asset price S, call strike K, time-to-exercise is T constant risk-free rate r, volatility σ the call option price is given by

$$\mathrm{BSformulaCall}(S,K,T,r,\sigma) = SN(d_1) - Ke^{-rT}N(d_2) \tag{20}$$

where N is the distribution function (cumulative) of the standard normal density and

$$d_1 = \frac{\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}\,.$$

It is sometimes convenient to define

$$P_1(S,K,T,r,\sigma) := N(d_1) \;\text{ and }\; P_2(S,K,T,r,\sigma) := N(d_2)\,. \tag{21}$$

With put-call parity we moreover have

$$\mathrm{BSformulaPut}(S,K,T,r,\sigma) = Ke^{-rT}(1 - P_2(S)) - S(1 - P_1(S))\,. \tag{22}$$

A.6 Historical volatility estimation for Black–Scholes model

Consider time-points $(t_i)_{i=0^N}$ and let $\Delta t_i := t_i - t_{i-1}$, $i = 1,\dots,N$. For the underlying risky asset we have

$$S_{t_i} = S_{t_{i-1}} \exp\left((\mu - \frac{1}{2}\sigma^2)(\Delta t_i) + \sigma(W_{t_i} - W_{t_{i-1}})\right).$$

Then (with $\stackrel{d}{=}$ denoting equal distributions)

$$\ln\frac{S_{t_i}}{S_{t_{i-1}}} \stackrel{d}{=} (\mu - \frac{1}{2}\sigma^2)(\Delta t_i) + \sigma\sqrt{\Delta t_i} Z_i)\,,$$

where Z_i are iid $N(0,1)$. Let $X_i := \ln\frac{S_{t_i}}{S_{t_{i-1}}}$. Then

$$\mathrm{Var}(X_i) = (\Delta t_i)\sigma^2 \mathrm{Var}(Z_i) = (\Delta t_i)\sigma^2\,.$$

And so

$$\sigma^2 = \operatorname{Var}\left(\frac{X_i}{\sqrt{\Delta t_i}}\right).$$

Thus, if $(s_i)_i$ are actual empirical observations of our time-series and at times t_i and we take $x_i := \ln \frac{s_i}{s_{i-1}}$, $\bar{x} := \frac{1}{N}\sum_{i=1}^N x_i$, we then have

$$\sigma^2 \approx \frac{1}{N-1}\sum_{i=1}^N \frac{(x_i-\bar{x})^2}{\Delta t_i}.$$

A.7 Fast Fourier Transform for Fourier Transform Approximation

We broadly follow [4, Ch. 11, Section 1] and [3, Section 4] and start by noting that FFT will efficiently evaluate

$$x(u) = \sum_{j=0}^{N-1} \exp\left(-i\frac{2\pi}{N}\, j\, u\right)\xi(j) \;\; \text{for } u = 0,\dots,N-1 \tag{23}$$

as long as N is a power of 2. This can be used to calculate

$$\mathbb{C} \ni k \mapsto f(k) = \frac{1}{2\pi}\int_{\mathbb{R}} \zeta(v)e^{-ivk}\,dv \;\in \mathbb{C}.$$

General case To approximate the integral take $A > 0$ and $N \in \mathbb{N}$ large so that with the grid $v_j := -A/2 + j\eta$, $\eta := A/(N-1)$ we have

$$\begin{aligned} f(k) &\approx \frac{1}{2\pi}\int_{-\frac{A}{2}}^{\frac{A}{2}} \zeta(v)e^{-ivk}\,dv \approx \frac{1}{2\pi}\sum_{j=0}^{N-1} w_j\zeta(v_j)e^{-iv_jk}\frac{A}{N} \\ &= \sum_{j=0}^{N-1} w_j\zeta\left(-\tfrac{A}{2}+j\eta\right)e^{-i\left(-\frac{A}{2}+j\eta\right)k}\frac{1}{2\pi}\frac{A}{N}, \end{aligned}$$

where w_j are weights for a given integration rule e.g. $w_0 = w_{N-1} = \frac{1}{2}$ and $w_j = 0$ for $j = 1,\dots,N-2$. Now take $k_u := \frac{2\pi}{N}\frac{1}{\eta}u$ so that

$$f(k_u) \approx \sum_{j=0}^{N-1} w_j\zeta\left(-\tfrac{A}{2}+j\eta\right)e^{-i\frac{2\pi}{N}ju}\frac{1}{2\pi}\frac{A}{N}e^{\frac{A}{2}\frac{2\pi}{N}\frac{1}{\eta}u}.$$

Now let

$$\xi(j) := w_j\zeta\left(-\tfrac{A}{2}+j\eta\right)\frac{1}{2\pi}\frac{A}{N}e^{\frac{A}{2}\frac{2\pi}{N}\frac{1}{\eta}u}.$$

Then

$$f(k_u) \approx \sum_{j=0}^{N-1}\xi(j)e^{-i\frac{2\pi}{N}ju},$$

which is exactly in the form (23).

Method for the case then the transform is real In this case we are more interested in evaluating

$$\mathbb{R} \ni k \mapsto f(k) = \frac{1}{\pi}\int_0^\infty \zeta(v)e^{-ivk}\,dv \;\in \mathbb{R}.$$

To approximate this we consider $\eta > 0$ defining the grid $v_j := \eta j$, $j = 0,\dots,N-1$. Then

$$f(k) \approx \frac{1}{\pi}\sum_{j=0}^{N-1}\zeta(v_j)e^{-iv_jk}\,\eta \tag{24}$$

The error arises from integrating only from 0 to $N\eta$ and from taking a piecewise constant approximation of the integrand.

We now need to discretize the target space: fix $\lambda > 0$ and consider the grid $k_u := -b + \lambda u$ for $u = 0, \ldots, N-1$. This gives us log-strike levels from $-b$ to $b = \frac{1}{2}N\lambda$. Substituting this and the expression for v_j into (24) we arrive at

$$f(k_u) \approx \frac{\eta}{\pi}\sum_{j=0}^{N-1} \zeta(v_j)e^{-i\eta j(-b+\lambda u)} = \frac{\eta}{\pi}\sum_{j=0}^{N-1} e^{-i\eta j\lambda u}\zeta(\eta j)e^{ib\eta j} \text{ for } u = 0, \ldots, N-1\,.$$

Let $\eta\lambda = \frac{2\pi}{N}$ and let

$$\xi(j) = \frac{\eta}{\pi}\zeta(\eta j)e^{i\,b\eta j} \text{ for } j = 0, \ldots, N-1\,.$$

Then

$$f(k_u) \approx \sum_{j=0}^{N-1} e^{-i\frac{2\Pi}{N}\,j\,u}\xi(j) \text{ for } u = 0, \ldots, N-1$$

ie a direct application of the discrete Fourier transform.

Note that [3] propose to improve accuracy with the use of weights coming from Simpson's rule:

$$\xi(j) = \frac{\eta}{\pi}\zeta(\eta j)e^{i\,b\eta j}\frac{1}{3}[3 + (-1)^{j+1} - \delta_j] \text{ for } j = 0, \ldots, N-1\,,$$

where $\delta_j = 1$ for $j = 0$ and 0 otherwise.

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